(Forthcoming in O. Rechter & C. Posy, eds., *Kant's Philosophy of Mathematics, Vol. II*, Oxford University Press, 2020/21. Please do not cite this 2018 draft in publications!)

From Dedekind to Cassirer: Logicism and the Kantian Heritage

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Logicism in the philosophy of mathematics is usually seen as one of the main reactions against Kant's claim that mathematical knowledge is synthetic *a priori*. Typically Frege and Russell are taken to be its main representatives. At the core of the present essay will, however, be Richard Dedekind. Dedekind is sometimes mentioned as an early logicist too, but not always, and we will consider reasons for doubts in this context as well. One major philosopher who took Dedekind to be a logicist and, strikingly, considered his relevant contributions to be superior to Frege's and Russell's is Ernst Cassirer. Indeed, Cassirer adopted Dedekind's logicism, which he, in another striking twist, took to be compatible with basic Kantian commitments. Our second core theme will thus be Cassirer's attempt to "logicize" Kant along Dedekindian lines. This attempt underwent subtle changes in Cassirer's later writings, by reemphasizing some Kantian dimensions.

The essay is structured as follows. In Section 1, a brief reminder about Dedekind's main contributions to the foundations of mathematics will be provided. Section 2 will deepen the discussion of Dedekind's logicism, based on his explicitly stated motivation for it. In Section 3, we will consider several criticisms of his approach, together with some initial, partial responses to them, thereby acknowledging certain lacunae in his approach. Section 4 will bring into play Cassirer's sympathetic reception of Dedekind's position, from his early writings on, which he interprets as a form of logicist structuralism. In Section 5, we will address how that reception fits into the "logical idealism" of the Marburg School, of which Cassirer was a member. In Section 6, his views on the role of intuition in mathematics will be reconsidered, thus probing the relationship between logicism and the Kantian heritage more deeply. The paper will close with a brief summary and conclusion.

1. Dedekind's Contributions to the Foundations of Mathematics

Dedekind was one of the most creative and influential mathematicians of the nineteenth century. Among historians, he is celebrated for his trailblazing work in algebra and algebraic number theory, especially his theory of ideals.[1] Among philosophers, his two booklets on the foundations of mathematics are more widely known: *Stetigkeit und irrationale Zahlen* (1872) and *Was sind und was sollen die Zahlen?* (1888).[2] The former concerns the real numbers, constructed out of the rationals as a continuous ordered field, while the latter introduces the natural numbers, based on the notion of a simple infinity. It is in the latter where Dedekind also presents his logicist convictions most explicitly, together with the results on which they are based. Let us briefly review the procedures in both texts.

In his 1872 essay, Dedekind starts by comparing the intuitively given geometric line with the system of rational numbers, conceived of as an ordered field. Putting aside geometric intuitions, he then defines what it means for the rationals to be densely ordered. He also distinguishes such denseness from continuity (line completeness) by means of his notion of a cut ("Dedekind cut"); and he points out that the system of rationals is not continuous, i.e., not every cut in it is determined by a rational number (e.g., the cut corresponding to $x^2 \leq 2$). Next he considers the set of all cuts on the rational numbers, endowed with operations of addition and multiplication and an ordering, all induced by corresponding features of the system of rationals. Most centrally, he shows that what results is an ordered field that is continuous with respect to its ordering, and one into which the system of rational numbers can be embedded naturally (via a homomorphism for ordered fields).

All the results in Dedekind's 1872 essay as just surveyed have become standard ingredients in later presentations of the (classical) foundations of real analysis and, most explicitly, in their set-theoretic reconstruction. Within set theory, the next

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[1] For an overview of Dedekind's mathematical work, with emphasis on their close relation to his more foundational contributions, see Reck (2016).

[2] In English: "Continuity and Irrational Numbers" and "The Nature and Meaning of Numbers" (or: "What are numbers and what are they for?"); cf. Dedekind (1872, 1888).

step is to say that we can take the real numbers to "be" the constructed cuts, i.e., to identify the ordered field of real numbers with the constructed system of cuts for most mathematical purposes. But this is not what Dedekind does. Instead he adds a further step, by appealing to the notion "free creation":

> Whenever, then, we have to do with a cut (A_1, A_2) produced by no rational number, we create a new, an *irrational* number α, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number α corresponds to this cut, or that it produces this cut. (Dedekind 1963, p. 15, original emphasis)

What such remarks suggest is that Dedekind introduces "the system of real numbers" as a separate continuous number field, isomorphic to the constructed system of cuts but distinct from it. He does not elaborate further on the notion of "creation" in his 1872 essay; but he will come back to it in his 1888 essay.

The main goal of Dedekind's 1872 essay is to provide a systematic introduction of the real numbers, based on the rational numbers and certain set-theoretic constructions. As such, it is a contribution to the "arithmetization of analysis" in the nineteenth century. "Arithmetization" implies that all we need for analysis are the natural numbers together with some implicitly presupposed "laws of thought". Dedekind was well aware that the rational numbers can be constructed out of the integers and, in turn, the integers out of the natural numbers, in each case as (equivalence classes of) pairs, thus completing such an "arithmetization".[3] But then, what about the natural numbers themselves, i.e., their "nature and function"? Also, what about the set-theoretic constructions performed along the way? This is what Dedekind's 1888 essay is about. In fact, it is an attempt to show that one can push the arithmetization of analysis a step further, to a "logization".

In that essay, Dedekind starts with the notions of object ("Ding"), set ("System"), and function ("Abbildung"), each of which he considers to be "logical". Within such a framework, he defines the notion of an infinite set ("Dedekind-infinite"), and then the notion of a "simply infinite set" (basically a model of the "Dedekind-Peano axioms"). In his *Theorem 66*—the most controversial part of the essay—he also

[3] Cf. Sieg & Schlimm (2005) for details and corresponding archival evidence.

tries to prove the existence of infinite sets, thus of simple infinities. For that purpose, he appeals to "the totality S of all things which can be objects of my thought"; to his "self" or "ego" as an element of that totality; and to the function ϕ on S that maps x to "x can be the object of my thoughts", which plays the role of a successor function (Dedekind 1963, p. 64). His argument is then that the set constructed by closing the singleton {his self} under ϕ (the corresponding "chain") forms a simply infinite subset N of S. Later Dedekind establishes that all simple infinities are isomorphic, i.e., mappable onto each other by 1-1 functions that preserve the order induced by each system's successor function.

In current axiomatic set theory we do not appeal to any "self", nor do we work with a function that has "thoughts" as arguments or values. Instead, we take the empty set, $\emptyset$, as our starting point, i.e., as (playing the role of) the number 0; and we use either $s: x \rightarrow \{x\}$, along Zermelo's lines, or $s': x \rightarrow x \cup \{x\}$, along von Neumann's lines, as our successor function. But Zermelo's procedure, in particular, is quite close to Dedekind's (with the small difference that he begins with the number 0, not with 1).[4] Dedekind does, however, not identify the simple infinity N he constructed initially with "the natural numbers"; nor would he want to do so with either Zermelo's or von Neumann's system, as is standard nowadays. Instead, here we encounter another appeal to "creation", at this point spelled out further in terms of the notion of "abstraction". As Dedekind writes (*Definition 73*):

> If in the consideration of a simply infinite system N set in order by a function ϕ we entirely neglect the special character of the elements, merely retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting function ϕ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series N*. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind (*ibid*., p. 68, original emphasis).

Later in the essay, after his proof that any two simple infinities are isomorphic, Dedekind justifies this introduction of the natural numbers further by observing

[4] A main exception is that axiomatic set theory does not allow for a "universal set", like in Dedekind, because of Russell's antinomy. We will come back to this aspect in Section 3.

that any theorems that holds for one simple infinity "possesses perfectly general validity" for any other (*Remark 134*, pp. 95-6). In that sense, his procedure is invariant under the choice of the simple infinity at the start. What Dedekind outlines, in other words, is a structuralist conception of numbers.[5]

There are other details in Dedekind's 1888 essay that are noteworthy and relevant for our purposes. For example, he shows how his approach allows for a proof of the principle of mathematical induction; thus we do not have to assume it as a basic, non-logical principle for the natural numbers. His approach allows for a general, systematic justification of recursive definitions, which Dedekind applies to his recursive definitions for addition and multiplication based on the successor function. And towards the end of the essay he points out how initial segments of his number series can be used as tallies for measuring the cardinality of finite sets, i.e., sets that are not Dedekind-infinite. This establishes that the "ordinal numbers" he has introduced can play the role of "cardinal numbers" too; and it leads to an alternative characterization for the finitude of sets.

Dedekind's two essays constitute "foundational" investigations, i.e. systematic studies of the notions and principles on which arithmetic—in the broad sense, from the natural to the real numbers (and implicitly even the complex numbers)—can and should be based. They are also meant to establish that arithmetic is "a part of logic". But more needs to be said about what exactly that means for him.

2. Dedekind's Logicist Project and its Motivation

Generally speaking, logicism is the thesis that all of mathematics, or at least core parts of it, can be "reduced to logic". This is usually taken to combine two sub-theses: (i) that all mathematical concepts can be defined in terms of logical concepts; (ii) that all mathematical truths can be derived from logical truths via logical inferences. As a response to Kant, but also to empiricists like J.S. Mill's, this is typically taken to imply: (iii) that we do not need to appeal to intuition in either

[5] See Reck (2003) for a more fully developed account of Dedekind's structuralism; but compare Sieg & Morris (forthcoming) for an alternative structuralist interpretation.

connection, i.e., in the relevant definitions and derivations (neither in the form of Kantian "pure intuition" nor sense perception). Kant saw such a need since for him mathematics relies on an intuitive "construction of concepts", e.g., when we construct triangles, circles, etc. in Euclidean geometry (an aspect that goes beyond traditional Aristotelian logic). He added to this an account of the intuitive forms of space and time as conditions for the possibility of all experience.

In the present essay, logicism will be understood as defined by conditions (i) and (ii). Its supposed consequence (iii) will play a role too, but in subtle, eventually problematized ways. Sometimes further claims are associated with logicism, such as: (iv) that reducing mathematics to logic shows it to be "analytic"; or (v) that doing so establishes its "certainty", thus establishing "foundations" for mathematics in a strong sense. For us, neither (iv) nor (v) are necessary for logicism. In fact, Frege and Russell—the authors taken to be its paradigmatic representatives—disagree on these points. Frege's goal was explicitly to establish the "analyticity" of mathematics by reducing it to logic (at least in his early writings, while later this becomes less central); Russell took that reduction to prove that mathematics is "synthetic", because he assumed logic to be synthetic. And while for Russell certainty is often an important goal, it does not loom as large for Frege.[6]

Unlike Frege and Russell, Dedekind does not prominently refer to Kant, much less to Mill, in his foundational essays. Nevertheless, he opposes the view that intuition is necessary for arithmetic, thus subscribing to claim (iii). This is most explicit in the Preface to Dedekind's 1888 essay, where he talks about his goal of developing "that part of logic which deals with the theory of numbers". He adds:

> In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number concept entirely independent of the notion of intuition of space and time, that I consider it an immediate result from the laws of thought (Dedekind 1963, p. 31).

The direct connection drawn here between "intuition" and "space and time" makes

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[6] To complicate things further, Frege and Russell differ significantly in their understanding of the notions of "analytic"/"synthetic" and "certainty"; cf. Kremer (2006) and Reck (2013a).

it hard to believe that Dedekind does not have Kant, or Kantian views, in mind as his foil. We also know that he was familiar with Kant's philosophy from a lecture course by Hermann Lotze, which he attended as a student in Göttingen.[7] Moreover, various nineteenth-century philosophers of mathematics, such as Sir William Hamilton (to whom Mill reacted in a well-known polemic), defended a Kantian appeal to intuition. Dedekind may have been familiar with some of those as well.

Nevertheless, the main way in which Dedekind motivates his logicist project is not by taking sides in an inner-philosophical debate. Instead, he ties it to questions about mathematical methodology. This is clear already in his 1872 essay. As noted above, it provides a systematic treatment of the real numbers based on the rationals and certain "logical" constructions. How had mathematicians proceeded before; and why was Dedekind not satisfied with their procedures? His answer is instructive:

> [T]he way in which the irrational numbers are usually introduced is based directly upon the conception of extensive magnitude—which is nowhere carefully defined—and explains number as the result of measuring such a magnitude by another of the same kind. (Dedekind 1963, pp. 9-10)

Here Dedekind is referring to the traditional treatment of magnitudes, and ratios of them, that goes back to Euclid.[8] His basic complaint is that the notion of magnitude is "nowhere carefully defined". There is also a deeper problem: ratios of them "can be clearly developed only after the introduction of irrational numbers" (p. 10, fn.*). In other words, the notion of irrational number, or of real number more generally, has to be in place for giving a clear account of ratios, not *vice versa*.

On what basis did mathematicians reason about magnitudes, their ratios, etc., if not by starting from an explicitly defined concept of real number? As Dedekind notes, they had "recourse to geometric evidences" (p. 1). There are various problems with such recourse, which he illustrates with two results from his 1872 essay. The first concerns how to treat the multiplication of irrational numbers, even relatively

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[7] Lotze's class, held in the summer semester of 1852, was called "Geschichte der neueren deutschen Philosophie seit Kant" ("History of Recent German Philosophy since Kant"). Notes from it are preserved in the Dedekind *Nachlass* in Göttingen.

[8] Cf. Stein (1990) and Muller (2006), especially Ch. 3.

simple ones such as square roots. Traditionally the way to deal with such numbers, or with corresponding "incommensurable magnitudes", was geometrically. We can construct $\sqrt{2}$ as the diagonal of a unit square; we can also multiply $\sqrt{2}$ and $\sqrt{3}$ by constructing a resulting magnitude geometrically. But how do we prove, in full generality, that $\sqrt{n}\,\sqrt{m} = \sqrt{nm}$, or even, $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ for rational numbers a and b? No such proof had been given before Dedekind's essay. He then provides one based on his notion of a cut. His approach also allows for a general, uniform construction of all irrational numbers, another basic ingredient missing up to then.

The second relevant result from Dedekind's 1872 essay is a central theorem in real analysis (the Calculus): the Mean Value Theorem. Consider a continuous function f that takes value a for argument x and value b for argument y. The theorem at issue says that, for any c between a and b (any intermediate or "mean" value) there must be a z between x and y such that $f(z) = c$. Intuitive this appears obvious (consider drawing the graph of the function), and for a long time that was taken to be sufficient evidence for the theorem. But again, can it be proven explicitly and more precisely? It can—but only if we have an adequate definition of continuity for functions, as provided by Cauchy, Weierstrass, etc., together with an adequate definition of the continuity of the real number line, as added by Dedekind.

Probed further, why is the recourse to intuitive properties of figures in space and time, as studied in Euclidean geometry, unsatisfactory in such contexts? Partly this is because the mere appeal to them obscures crucial distinctions, e.g., that between denseness and continuity. As emphasized by Dedekind, unaided spatio-temporal intuition is too coarse-grained and fuzzy, hence in need of help itself. And he adds: "It is only through the purely logical process of building up the science of numbers and by thus acquiring the continuous number domain that we are prepared to accurately investigate our notions of space and time" (p. 37). The basic issue here, similarly to above, is that a precise, systematic, study of space and time needs arithmetic and logical notions, not *vice versa*. Dedekind brings this point home by noting that all of Euclidean geometry holds in a space of points corresponding only to algebraic numbers, thus one that is dense but not continuous (p. 38).

If we do not want to appeal to traditional geometry, or to the intuitive evidence associated with it, as the foundation for arithmetic, what should we use instead? Dedekind's answer is: concepts defined explicitly in terms of arithmetic and logical notions.[9] Moreover, the most basic notions for him, as we saw above, are those of thing, set, and function. It is these that underwrite his definitions of continuity, real number, infinity, natural number, etc. This is the core of Dedekind's logicism; thus he clearly subscribes to our condition (i) above. And what kind of principles or basic laws are we meant to use in the relevant derivations? Only general "laws of thought", as he sometimes put it; hence we find condition (ii) as well. Having said that, Dedekind does not make the needed basic laws explicit, only his basic notions and corresponding definitions. (We will come back to that point below.)

For present purposes, one more detail in this general context is worth highlighting. Among Dedekind's basic logical notions, to be used in his logicist reconstruction of the theory of the natural numbers, he highlights one especially:

> If we scrutinize closely what is done in counting an aggregate or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible at all (Dedekind 1963, p. 32).

One striking aspect of this passage is that Dedekind ties the notion of function to a fundamental "ability of the mind". Now, are there such "abilities" concerning the notions of thing and set for him as well; or is the "functional ability" sufficient in itself? Unfortunately, he does not address those questions anywhere.

3. Criticisms and Partial Defenses of Dedekind's Logicism

As we saw, Dedekind's aim is to establish that arithmetic, understood in the broad sense, is "a part of logic". And this has both a conceptual and a deductive side, in the sense of our characteristics (i)-(ii) for logicism. In addition, his results are expressly meant to have "anti-intuitive" consequences, corresponding to (iii). Moreover, after

[9] Cf. Klev (2011) for Dedekind's focus on concepts, definitions, and what can be derived from them. With the implicit emphasis on "conceptual reasoning", Dedekind stands in the tradition of Gauss, Dirichlet, and Riemann; cf. Stein (1988) and Reck (2013a).

Dedekind encountered Frege's attempt to reduce the theories of the natural and real numbers to logic, his reaction was: "[Frege] stands upon the same ground with me" (Dedekind 1963, p. 43, in the Preface to the 2nd ed. of his 1888 essay, published in 1893). After studying his 1888 essay carefully, Frege remarked in turn: "Dedekind too is of the opinion that the theory of numbers is a part of logic" (Frege 1893, p. 196). Several other contemporaries of theirs took Dedekind to be a logicist as well. To mention just one example, Ernst Schröder—a member of the Boolean school in logic and one of his most sympathetic early readers—wrote about the temptation to join "those who, like Dedekind, consider arithmetic a branch of logic".[10]

It seems hard to deny, then, that Dedekind should be considered a committed logicist, parallel to Frege and Russell (at least with respect to arithmetic[11]). To be sure, he does not subscribe to claims (iv) and (v), concerning "analyticity" and "certainty" (as far as one can tell from his silence on these topics). But we already noted that they should not be seen as necessary for logicism. Even putting such considerations aside, however, there are influential philosophers who have raised doubts about Dedekind's logicism, including Russell, more recently George Boolos and Michael Dummett. These critics claim that Dedekind's position cannot be seen as a successful form of logicism, or they argue that there are problems with it that make it easy to dismiss his philosophical views more generally. We will soon provide some initial, partial responses to such criticisms. Yet it has to be admitted that Dedekind's discussion is incomplete, at the least, in certain respects.[12]

The most important technical objection to Dedekind's project comes from Russell. Russell was aware of Dedekind's foundational work from early on, as is evident

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[10] C.S. Peirce and D. Hilbert were two other contemporaries who considered Dedekind a logicist; cf. Ferreirós (1999, 2009) and Reck (2013a). For later acknowledgments of him as a logicist, see Stein (1998), Demopoulos & Clarke (2007), and Klev (2017).

[11] While Dedekind's remarks about space suggest that he would approach geometry from a "logical" point of view as well, we are not aware of any place where he worked this out in detail. We thus focus on arithmetic in the present paper (similarly for Cassirer below).

[12] Cf. Benis-Sinaceur (2015) for an interesting, somewhat different challenge to seeing Dedekind as a logicist. This recent challenge deserves a detailed response, which cannot be provided in the present essay. Cf. Ferreirós (forthcoming) for an initial reaction.

from his *Principles of Mathematics* of 1903. He was more enthusiastic about Frege's and Georg Cantor's works, thus more elaborate in his responses to them. Yet one of his main discoveries, of which he informed Frege in 1902, affects Dedekind too: Russell's antinomy. Recall that Dedekind appeals to "the totality *S* of all things which can be objects of my thought". It is natural to interpret this as implying his commitment to an unconstrained comprehension principle for sets, which falls pray to Russell's antinomy. Dedekind found out about closely related problems not from Russell but from Cantor, who informed him already in 1899 that the "collection" of all sets, thus also the bigger "collection" of all things, is an "inconsistent totality". His initial reaction, like Frege's to Russell's antinomy, was shock; but later he convinced himself that it must be possible to find a way around it.[13]

Russell's antinomy is often taken to undermine approaches like Dedekind's decisively, by showing that they cannot be developed consistently. Yet it is not, in itself, a reason to think that he did not pursue a logicist project. After all, the antinomy also affects Frege, one of our paradigmatic logicists. But two other considerations by Russell are meant to go further. First, he is skeptical of Dedekind's structuralist conception of mathematical objects, asserting that "if [numbers] are to be anything at all, they must be intrinsically something" (Russell 1903, p. 203). Related to that, he insists that Dedekind's appeal to "abstraction" should be replaced by his own "principle of abstraction", i.e., the use of equivalence classes in defining numbers. Second, Russell takes Dedekind's appeal to "thoughts" in his proof of Theorem 66 to be objectionable. One might think that he takes it to make the approach problematically psychologistic; but that is not the case. Instead, Russell's objection is that Dedekind brings in notions that "are not appropriate to mathematics", much less to logic (Russell 1904, p. 258).

Both of these more philosophical criticisms of Dedekind have produced strong echoes in the literature. Picking up on Russell's second point, Boolos calls

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[13] For Dedekind's initial reaction, see fn. 9 in Reck (2003). His considered response can be found in the 1911 Preface to the 3rd edition of *Was sind und was sollen die Zahlen?*, where he assures his readers and himself of the "inner harmony" of his approach.

Dedekind's proof of *Theorem 66* "one of the strangest pieces of argumentation in the history of logic", since it appeals to "as wildly non-mathematical an idea as his own ego" (Boolos 1998, p. 202). In line with Russell's first argument, Dummett writes: "[Dedekind] believed that the magical operation of abstraction can provide us with specific objects having only structural properties: Russell did not understand that belief because, very rightly, he had no faith in abstraction thus understood" (Dummett 1991, p. 52). Dummett also takes the kind of abstraction at play in Dedekind to involve psychologistic aspects, in the sense of involving a mental process of "creation", or even the creation of "mental objects", an idea that is brushed aside thus: "Frege devoted a lengthy section of *Grundlagen*, sections 29-44, to a detailed and conclusive criticism of this misbegotten theory" (p. 50).

While these criticisms are uncharitable and much too dismissive, Dedekind opens himself up to them by talking, unguardedly and without further elaboration, about "thoughts", his "self" or "ego", the "creation" of numbers, and "abilities of the mind". Then again, several immediate defenses of Dedekind are not hard to provide. To begin with, his appeal to "thoughts" in Theorem 66 need not be taken in Dummett's crude psychologistic sense. Instead Dedekindian "thoughts" may be understood in a more objective sense, akin to Frege's notion (as both Frege and Russell did in their responses).[14] Similarly, his remarks about "the mind", "mental abilities", etc. might, and arguably should, be taken in a less individualistic, more objective sense too.[15] More directly, note that Dedekind does not talk about "the totality *S* of objects of my thought", but instead, "the totality *S* of all things which can be objects of my thought". Thus he does not appeal to the contents of his individual "mind" anyway.

Another response to Russell, Boolos, and Dummett takes us back to the fact that it is not hard to replace Dedekind's appeal to a sequence of "thoughts" and his "self" by, say, Zermelo's use of $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, … . In fact, making use of Zermelo's suggestion

[14] Dedekind's proof of the existence of an infinite set is also similar to an argument by Bolzano, which involves a parallel objective notion of "thought". Cf. Reck (2013a/b).

[15] The English translation of "Geist" as "mind" is misleading here. "Geist" and "geistig" as used in nineteenth century Germany (cf. "Geisteswissenschaften" etc.) suggest a socio-cultural rootedness, thus inter-subjectivity, not captured by "mind" in English.

seems congenial to Dedekind's general employment of sets, even if this particular construction did not occur to him.[16] Finally, while he fails to formulate basic laws either for his set-theoretic constructions or for the abstraction to which he appeals, it might be possible to provide such laws for him, instead of simply dismissing his approach as psychologistic. And on that basis, we might try to discuss in a more charitable and substantive way whether they should count as logical or not.

This brings us back to Frege's response to Dedekind. As we saw, Frege views him as pursuing a logicist project. However, he does not take that project to be successful, in the sense of actually establishing logicism. There are three main points Frege makes in this context.[17] First, he suggests that Dedekind's use of the notions of "set", "element", etc. raises doubts about whether these are logical notions. Second, he notes that the proofs in Dedekind's foundational essays are too sketchy to be sure that intuition has not crept in somewhere. Third, Frege highlights the fact that Dedekind has not explicitly formulated the basic laws on which his project rests. I take this third point to be the most serious criticism by Frege, since the other two depend on it. Not only is it impossible to check whether Dedekind's approach is consistent without having its basis open to inspection. It is also hard to make sure the "gaps" in his proofs can be filled in appropriately; and we cannot determine decisively whether his notion of "set" should be taken to be logical or not.

In themselves, Frege's criticisms do not establish that Dedekind's project fails, only that it is incomplete. Might it be possible to complete it for him, by supplementing the basic laws he needs? This is a difficult question, especially if the result is supposed to allow for responses to all the questions raised so far. What would be required are laws that satisfy four conditions: (i) They are in line with, or at least in the spirit of, his overall approach. (ii) They add up to a consistent system, or at least one that does not obviously fall pray to Russell's and similar antinomies. (iii) The laws, and thus the approach, are arguably logical. Finally: (iv) We need laws that

[16] Dedekind does not introduce the empty set in his 1888 essay, because he thinks he has no need for it. But he acknowledges that it could be introduced for other purposes.

[17] For further details concerning Frege's reaction to Dedekind, see Reck (forthcoming a).

underwrite not only the set-theoretic constructions Dedekind employs, but also the kind of abstraction on which his structuralism depends.

Let us look at the construction side first, briefly. Which set-theoretic constructions are needed along Dedekind's lines? Also, does he provide any hints about the form of corresponding laws? Concerning the first question: We need to be able to form cuts on the system of rational numbers, and then form the system of all such cuts, to get to the real numbers. Similarly, we need pairs of numbers, equivalences classes of such pairs, and systems of all of them, to get from the natural numbers to the integers and rationals. Finally, we need an infinite sequence of objects, such as ∅, {∅}, {{∅}}, … , plus the set containing all members of that sequence, for the natural numbers.[18] With respect to the form of corresponding laws: Dedekind is explicit about using an extensional notion of set, even if he does not formulate it as an axiom. But should we really take him to assume an unrestricted comprehension principle? Closer to Dedekind's actual practice is the following: He appeals to a universal set together with a general subset (or separation) principle. We have learned, of course, that this will not work on pain of inconsistency.[19]

As mentioned earlier, Dedekind was shocked about the set-theoretic antinomies initially, while later he expressed confidence that a solution would be found. In fact, late in his life he became interested in the work of Ernst Schröder, who had not only adopted many of Dedekind's techniques (e.g. the notion of "chain"), but also started to introduce a general theory of sets or classes.[20] In addition, one may wonder whether Dedekind became aware of Zermelo's efforts to axiomatize set theory after 1900. (We are not aware of any evidence for it.) Zermelo was strongly influenced by Dedekind as well. Moreover, axiomatic set theory allows for all the constructions

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[18] In set-theoretic terms, Dedekind's laws need to include, or at least imply: the powerset axiom, an axiom for Cartesian products, and the axiom of infinity. In addition, the axiom of replacement is needed, e.g., for his general treatment of recursion; cf. Kanamori (2010).

[19] Cf. Ferreirós (forthcoming) for more on this point. Assuming the existence of a universal set is not inconsistent in itself, of course; but we then have to be very careful about which other principles to add. In particular, it is incompatible with a general subset principle.

[20] Cf. Schröder (1890); see Ferreirós (1999, forthcoming) for more.

Dedekind needs. Yet another option, along different lines, would be to turn to category theory, which also allows for all the needed constructions. In that respect, Dedekind's emphasis on the notion of function, or morphism, is intriguing.

The axioms of set theory are usually not considered logical laws today; even less so for basic category-theoretic principles. That would seem to rule out both as candidates for completing Dedekind's logicist project. Then again, it is not entirely clear why the set-theoretic axioms should not be seen as "logical". Remember that we set the notions of "analyticity" and "certainty" aside in terms of characterizing logicism. Hence, if the ZFC axioms, say, are to be disqualified as "logical", it has to be on other grounds. Moreover, note that what "logicism" amounts to in the present context is basically the attempt to found arithmetic, as well as other parts of mathematics, on a general theory of sets and functions. From that point of view, it is hard to deny that Dedekind was a logicist. But it leads to the question of whether the notion of "logic" has then been watered down too much, or whether it is then used in an unprincipled and *ad hoc* way that is philosophically toothless.

In the last few paragraphs, we considered possible Dedekindian "construction principles". But what about principles for his structuralist abstraction? That side may seem even more problematic. Not only did Dedekind not formulate basic principles for abstraction either; his few relevant remarks are often taken to be psychologistic. What would need to be done, then, is to formulate explicit, clearly non-psychologistic, and arguably logical principles for "Dedekind abstraction".[21] This is a subtle, technical issue that cannot be treated fully here. But there is one approach to it that can at least be mentioned. Namely, Øystein Linnebo and Richard Pettigrew have recently explored relevant abstraction principles roughly along along neo-Fregean lines. So far, their results are limited mathematically. Still, their approach, or some variant of it, may be sufficient for Dedekind's purposes.[22]

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[21] The label 'Dedekind abstraction' comes from Tait (1997), as does the suggestion to see it as a logical procedure. Compare Reck (2003, forthcoming b) for more.

[22] Cf. Linnebo & Pettigrew (2014). A more general approach opened up along such lines is

4. Cassirer's Sympathetic Reception of Dedekind's Logicism

We saw above that Dedekind's argument for his *Theorem 66*, in terms of "the self" and "thoughts", together with his remarks about "mental abilities", "the mind", etc., have been criticized sharply. They have received a more positive reception as well, however, and one that involves interpreting him as a Kantian in certain respects. Thus, Philip Kitcher and David McCarty have both suggested readings according to which Dedekind is appealing to such notions in a Kantian transcendental sense.[23] This is meant as a defense of Dedekind, including undercutting the criticism that his account is psychologistic in a crude empiricist, individualistic, or subjectivist sense. However, it is not clear how such readings gets us around other objections, e.g., the Russellian complaint that he imports non-mathematical ideas into the foundations of mathematics. It does seem to bring in non-logical ideas, doesn't it? Or what would be a notion of "logic" according to which this is not the case?

While intriguing, Kitcher's and McCarty's suggestions are thus not satisfactory; at the least, they are underdeveloped.[24] But another proposal for how to synthesize Kant and Dedekind has been around much longer and is arguably more promising. We are speaking here of Ernst Cassirer's neo-Kantian reception of Dedekind's ideas. The earliest relevant works in this connection are: Cassirer's essay, "Kant und die moderne Mathematik" (1907), and his first systematic book, *Substanzbegriff und Funktionsbegriff* (1910).[25] Given our discussion so far, several points made in those early works are worth reconsidering right away: Cassirer's defense of Dedekind against Russellian and similar objections; his adoption of Dedekind's structuralism; and most strikingly, his appropriation of Dedekind's logicism. (As we will see in later sections, Dedekind keeps coming up in Cassirer's later writings, and in ways

to use neo-logicist principles like "Hume's Principle for the "constructive" side in Dedekind, and principles such as those explored by Linnebo & Pettigrew for the "abstraction" side.

[23] See McCarty (1995), earlier Kitcher (1986).

[24] Kitcher's understanding of Kantian transcendental conditions of experience is also avowedly psychologistic, thus re-opening that whole issue again.

[25] Cf. Cassirer (1907) and Cassirer (1910). As the former essay is still not translated into English, we will provide our own translations of passages to be quoted.

that complicates things, including with respect to the issue of logicism.)

Cassirer's 1907 essay on Kant is a response to Russell's *Principles of Mathematics*, from 1903. More proximately, it is also a response to Louis Couturat's *Les principes des mathématiques*, published in 1905. In the latter, Couturat adopts Russell's approach, but also attacks Kant's views about mathematics in a lengthy appendix. Unlike most neo-Kantians, Cassirer is open to Russell's new logic, yet he wants to come to Kant's defense as well. His reaction to Couturat has three basic ingredients: He agrees that Russellian (and Fregean) logic provides a crucial improvement over traditional logic; he brings up Dedekind's foundational views too, since he sees them as superior; and he argues that Dedekind's approach is compatible with central Kantian commitments. In addition, Cassirer puts Dedekind's contributions into the context of a very general transformation of the mathematical sciences: the shift from a "substance-based" to a "function-based" perspective, with Dedekind's works seen as a paradigmatic embodiment of the latter.

Already in his 1907 essay, Cassirer goes out of his way to praise the "philosophical fruitfulness of mathematical logic" (Cassirer 1907, p. 38). With Russell, he thinks of it as a "general logic of relations". Such logic, together with new developments in mathematics, illustrates "the power and purity of conceptual thinking" (p. 39). Implicitly this points in a logicist direction already; but Cassirer goes further. Mathematics at the beginning of the twentieth century is no longer the "science of number and quantity", as it was seen traditionally. It has turned into the much more general study of "relationally" or "functionally" determined structures. And this has led to a new task: to identify the core concepts of various mathematical theories, since they provide "the necessary and sufficient conditions upon which certain domains of theorems can be founded completely" (p. 39). Crucially, modern logic forms the framework in which one can formulate such concepts. Cassirer also agrees with Russell that, along such lines, "logic and mathematics have been fused into a true, henceforth inseparable unity" (p. 40). Seen historically, he locates that fusion in the tradition of a (Cartesian and Leibnizian) *mathesis universalis*.

With respect to the theory of numbers, Cassirer points to Dedekind as having taken the crucial steps in formulating its "necessary and sufficient conditions". Or as he puts it *Substanzbegriff und Funktionsbegriff*, Dedekind has done the most to distill out "the logical foundations of the pure concept of number" (Cassirer 1910, p. 35).[26] He achieved this is by characterizing the natural numbers as "a sequence of elements connected by means of a certain order" (*ibid*.), thus by focusing on the ordinal conception of numbers. The result is what Russell calls a "progression". Russell's fundamental tool in characterizing progressions is the notion of relation. Dedekind went further, in Cassirer's eyes, by taking the notion of function to be even more basic. This means not only that the order in which the natural numbers stand is defined in terms of the successor function; rather, the very notion of relation has been "traced back to the more fundamental idea of 'functionality'" (Cassirer 1907, p. 43). The latter lies at the core of Cassirer's suggestion to see modern mathematical science as based on "function concepts", in contrast to older, outdated "substance concepts" that remain rooted in Aristotelian logic, together with the ontology and psychology that go with it.[27] Finally, it is precisely in this context that Cassirer highlights Dedekind's striking remark, quoted earlier, about "the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible at all" (*ibid*.).

When Cassirer turns to the real numbers, the degree to which he takes Dedekind's insights to be decisive becomes even clearer. As for the natural numbers, what is crucial for the reals is to characterize the order in which they stand explicitly and precisely. Dedekind does this by first distinguishing denseness from completeness and by then establishing that the latter is the core concept. Consequently, it is possible to found the theory of the real numbers "on the pure concept of relation, thus making it independent of any geometric consideration" (Cassirer 1907, p. 47). As Cassirer points out, the crucial innovation here is Dedekind's notion of cut. It is

[26] Like in our discussion of Dedekind above, we will focus on Cassirer's views about arithmetic in the present essay. Concerning geometry, cf. Heis (2011).

[27] While this is the core of Cassirer's notion of "function concept", his substance-function dichotomy is not easy to pin down generally. Cf. Heis (2014) for further discussion.

what allows for "the essential conceptual characterization" in this context; and thus, a return to "intuitive geometric relations between magnitudes" is no longer needed (p. 53). The philosophical significance of all of this is that "the pure and abstract concept of number extends its reach into a realm that was commonly assigned to sensual intuition" (*ibid*.); or even more strongly, "everything that remains out of reach for sensibility forever is achieved by mathematical concepts" (p. 55). Clearly Cassirer has absorbed Dedekind's logicist point that unaided intuition is too vague and fuzzy for the foundations of analysis, hence in need of conceptual help.

Besides endorsing the logicist side of Dedekind's position explicitly, Cassirer also notes the structuralism that goes with it. He contrast Dedekind's conception of the natural numbers favorably with the traditional view of them as "multitudes of units", dominant from Euclid into the nineteenth century, by writing:

> [Dedekind showed that] in order to provide a foundation for the whole of arithmetic, it is sufficient to define the number series simple as the succession of elements related to each other by means of a certain order—thereby thinking of the individual numbers, not as 'pluralities of units', but as characterized merely by the 'position' they occupy within the whole series (Cassirer 1907, p. 46).

With respect to the reals, and especially the irrational numbers, he put it thus:

> We thus see that, to get to the concept of irrational number, we do not need to consider the intuitive geometric relationships of magnitudes, but can reach this goal entirely within the arithmetic realm. A number, considered as part of a certain ordered system, consists of nothing more than a position (*ibid*., p. 49).

As this makes clear, Cassirer adopts a Dedekindian combination of logicism and structuralism for the real numbers too, thereby leaving behind the traditional appeal to geometric intuition connected with the notion of magnitude.

In endorsing Dedekind's approach, Cassirer is fully aware of Russell's criticism of his structuralist conception. Quoting the core passage from Russell's *Principles* almost verbatim, he defends Dedekind's approach to the natural numbers as follows:

> If the ordinal numbers are to be anything, they must—so it seems—have an 'inner' nature and character; they must be distinguished from other entities by some absolute 'mark', in the same way that points are different from instants, or tones from colors. But this objection mistakes the real aim and tendency of Dedekind's formation of concepts. What is at issue is just this: that there is a system of ideal

> objects whose content is exhausted in their mutual relations. The 'essence' of the numbers consists in nothing more than their positional value. (1910, p. 39)

Cassirer's response to Russell in this context consists of three related points. First, he suggests that Russell's criticism of Dedekind remains in line with a traditional "substance" perspective, where all objects need to be distinguished by "absolute marks". Second, Dedekind has overcome this older perspective and replaced it by a "functional" one, resulting in his structuralist conception of number. Third, Dedekind is on the right track here, not Russell, since a "functional", and thereby structuralist, perspective is the one appropriate for modern mathematics.

In addition, there is a defense of Dedekind's logicist structuralism, or structuralist logicism, against the charge of psychologism in Cassirer's 1910 book as well.[28] Thus, he clarifies and supports Dedekind's notion of abstraction as follows:

> [In Dedekind] abstraction has, then, the effect of a liberation; it means logical concentration on the relational system, while rejecting all psychological accompaniments that may force themselves into the subjective stream of consciousness, which form no constitutive moment of this system (ibid., p. 39).

Here Cassirer not only rejects the view that Dedekind abstraction has anything to do with the content of our "subjective stream of consciousness", he also suggests a "logical" interpretation of it. Elsewhere, he distinguishes an older form of "abstraction"—tied to the Aristotelian "substance" tradition and understood in subjective psychological terms (selective perception, focused attention, etc.)—from a newer form, precisely as represented by Dedekind's work. In his view, Dedekind abstraction is thus far from subjective psychological processes. Then again, he does not offer a reconstruction in terms of logical laws for it, as requested by Frege. As Cassirer is not a mathematical logician, this would be asking too much of him.

5. Intuition, Logical Idealism, and Kant's Construction of Concepts

Cassirer is known as a neo-Kantian, and more precisely, a member of the Marburg School of Neo-Kantianism. But how can a philosopher endorse logicism while still taking him- or herself to be faithful to Kant? More specifically, how can Cassirer's

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[28] See Yap (2017) for more on this topic.

rejection of a foundational role for intuition in mathematics, along Dedekindian lines, be squared with any form of Kantianism? This question about Cassirer needs to be addressed. An initial answer is that philosophers in the Marburg School, from Hermann Cohen on, insist on downplaying the role of intuition more generally, as has often been pointed out in the secondary literature. These philosophers deny that intuition is a separate "faculty" of knowledge; instead, it is subordinate to the "understanding", thus to "logic". And they are particularly critical of appeals to intuition in a "psychological" sense, thus rejecting a psychologistic reading of Kant. The resulting position is "logical idealism", as Cassirer frequently calls it. As we will see, this part of his philosophical heritage is relevant in our context; but the relation between "logic" and "intuition" in Cassirer is more complicated in the end.

Above we saw that Cassirer highlights the "power and purity of conceptual thinking" in modern mathematics, which has "extended its reach into a realm that was commonly assigned to sensual intuition"; he praises Dedekind for making the concept of real number "independent of any geometric consideration"; and for the natural numbers too, what is crucial is "logical concentration on the relational system". Overall, Dedekind has provided us with "the logical foundations of the pure concept of number". This sounds like a strong, unreserved endorsement of Dedekindian logicism; and in particular, it appears to leave little room for Kantian intuition. But we would like to urge some caution here. Clear are two points, we suggest: (i) Following Dedekind, Cassirer denies that intuitive considerations along traditional Euclidean lines play a foundational role in modern mathematics (e.g., appeals to intuitive evidence for the Mean Value Theorem). (ii) He also rejects, along standard Marburg lines, that intuition in a more general "psychological" sense does so (including any subjective sensual or proto-perceptual sense).

If there were nothing more to Kant's appeal to intuition in mathematics than that, Cassirer's Dedekindian position would be strongly anti-Kantian. It fact, it would be hard to see how it is a "neo-Kantian" position. But does Cassirer see no significant role for intuition concerning mathematics at all? While his logicist rhetoric, as quoted above, may make it appear that way initially, we want to suggest two alternatives in the rest of this paper. First, Cassirer does acknowledge a role for intuition in mathematics, but in a less "pure" and non-foundational sense. Second, it plays a different role for him as well, even in a foundational sense, but one that is different from (i) and (ii). The first point can be tied naturally to Cassirer's Marburg heritage, it is present already in his early writings, and we will elaborate it further in the present section. There is evidence for the second—more surprising and probably more contentious—point as well, at least in Cassirer's later writings, thus indicating a subtle development in his position. But he never separates it clearly from the first, which makes his views hard to pin down at various points. The latter is also why this point has not been recognized more so far, we believe.

As mentioned earlier, for the members of the Marburg School "intuition" should not be seen as an independent "faculty", but as subordinate to Kantian "understanding". In line with this conviction, Cassirer writes in his 1907 essay that intuition is "not the source of the logical and mathematical principles, but already involves them and only represents them concretely" (Cassirer 1907, p. 67). Note that this remark leaves room for some role for intuition, even if not a primary and independent one. However, what does Cassirer mean by a "concrete representation" of logical and mathematical principles, and which role is it meant to play? This requires further clarification. With respect to the claim that it is intuitive forms of time, in particular, that underlie arithmetic, Cassirer adds:

> Against the Kantian theory, it has to be emphasized that it is not the concrete form of temporal intuition that forms the ground of the concept of number, but that in it the purely logical concepts of consequence and order are already contained implicitly and embodied. (Cassirer 1907, p. 68, fn. 54)

Here a "concrete form of temporal intuition" is mentioned; but one may wonder again what Cassirer means by it. Still, this "concrete form" is again not rejected, just assigned a secondary, dependent role. Doing so allows Cassirer to hold on to what he takes to be a more basic Kantian insight. Namely, the "genuinely new and original result of the critique of reason" consists in the insight that "the functions of pure understanding become apparent as the preconditions of 'sensuality'" (p. 69). In other words, sensual intuition is always already informed logically.

In Kant, the "functions of pure understanding", or the corresponding "categories", are tied to Aristotelian logic. More than most Neo-Kantians, Cassirer is open to replacing Aristotelian logic by modern logic; and he wants to do so in the form of Russell's "logic of relations", as developed further along Dedekindian lines. We saw above that the latter involves "tracing the notion of relation back to the more fundamental idea of 'functionality'"—the core of Cassirer's "functional" point of view. And again, this is precisely why he highlights Dedekind's remark about the fundamental nature of our ability to think "functionally". Now, for both Dedekind and Cassirer that ability is one "without which no thinking is possible at all", i.e., it is a precondition for all thinking. What this suggests is that for Cassirer, and more implicitly for Dedekind too, the notion of function takes the place, or should be seen as an integral part, of Kant's "categories of the understanding". And for that reason it is a central logical notion, in a broadened, updated sense of "logic".[29]

Cassirer's shift from traditional Aristotelian to Russellian/Dedekindian logic has another relevant consequence, concerning the distinction between "analytic" and "synthetic" judgments. Kant's main characterization of that distinction is in terms of the Aristotelian "*S* is *P*" form of judgments: A judgment is analytic if the predicate *P* is contained in the subject *S*; otherwise it is synthetic.[30] Sometimes he characterizes the distinction also in terms of what follows from two laws of traditional logic, the laws of contradiction and of identity: A judgment is analytic if it follows from those laws alone; otherwise it is synthetic. And these two characterizations are supposed to coincide. As Cassirer indicates already in his 1907 essay, he wants to hold on to the analytic-synthetic distinction, but view it as independent from Aristotelian logic. But then, neither of these two characterizations is attractive. Instead, he suggests going back to three more fundamental Kantian idea: First, a judgment should be called "synthetic" if it involves a kind of "synthesis", where this does not have to be

[29] A similar analysis of Dedekind's logicism, viewed against a general Kantian background, is presented in Klev (2017). However, no reference to Cassirer is made there.

[30] It should be added that Kant's conception of logic is broader than suggested by the "*S* is *P*" schema. In particular, it involves the notion of (partial and complete) disjunction too, and thus, that of relation. This is relevant for Cassirer's adoption of a Russellian theory of relations along (Neo-)Kantian lines, although he does not make this point very explicit.

explained by means of the "*S* is *P*" schema, but more generally in terms of relating a "one" to a "many". Second, crucial for the synthesis at issue is that it advances knowledge; or in Cassirer's own words: "Every judgment that has any value for progress in science should be called synthetic with respect to its origin" (Cassirer 1907, p. 70). And third, characteristic for the relevant synthesis and the resulting progress is the "holding of a pure law-governed relation" (p. 72).

Arithmetic judgments, or arithmetic theorems, clearly do have "value for progress in science". They also involve "law-governed relations", as Dedekind's reconstruction of them has made evident nicely. But then, they are "synthetic" by Cassirer's lights. Moreover, the relevant laws are logical laws, as Dedekind established; and they should be seen as synthetic too. Actually, a fourth and closely related aspect crucial for the syntheticity of mathematics needs to be mentioned here as well. All basic mathematical concepts involve an "existential" aspect; or in Cassirer's words:

> Every explanation of a basic concept in mathematics includes […] an 'existential claim': insofar as it says, at the same time, that some 'object' falls under the defined concept, i.e., a uniquely determined content of thinking (Cassirer 1907, p. 73).

For Cassirer two paradigm cases are, once again, Dedekind's treatments of the natural and the real numbers. With respect to the reals, the relevant "existential claim" concerns the existence of a complete ordered field, justified by means of Dedekind's construction of the system of cuts on the rationals. For the natural numbers, his initial construction of a simple infinity plays a parallel role.

It is exactly such Dedekindian constructions, with their "existential" import, that allow Cassirer to come back to Kant. Namely, he wants to uphold the Kantian claim that mathematics essentially involves the "construction of concepts". As mentioned earlier, Kant himself based the latter on the procedures of Euclidean geometry, e.g., when we construct $\sqrt{2}$ as the diagonal of a square or π via the circumference of a circle. This was quite appropriate for the mathematics of Kant's time; but it ties the relevant point closely to pre-modern, geometry-based mathematics. Consequently, it seems outdated and irrelevant with the shift to modern mathematics. But for Cassirer, Kant's view has not been undermined; it needs to be retained, albeit in a modified form. Namely, we have to replace traditional Euclidean constructions by Dedekindian "logical" or set-theoretic ones. In fact, that replacement lies at the core of Cassirer's way of "logicizing" Kant, as can now be seen.

Now, is intuition not involved at all in such Dedekindian constructions, thus in the "synthesis" and "existential" aspect characteristic of mathematics? In Cassirer's 1907 essay and in his 1910 book, one can, once again, get the sense that he excludes intuition entirely. But we have started to question that interpretation already, by mentioning a secondary, derivative role for intuition acknowledged by him. But how is it connected with the Dedekindian constructions just invoked? Is this where "concrete representation" of logical and mathematical principles fit in, i.e., is their role to underwrite those constructions? But if so, why do they not play a rather fundamental role? This issue is left open, or at least ambiguous, in Cassirer's early writings we believe. The result is a tension in them—and arguably in Dedekind's contributions on which they rely—concerning the relationship between "logic" and "intuition". Moreover, this issue keeps reverberating in Cassirer's later writings, e.g., in his *Philosophie der Symbolischen Formen, Vols. 1-3*, published in the 1920s, and in Volume 4 of *Erkenntnisproblem*, written in the late 1930s.

6. Cassirer's Later Writings and his Developing Views about Intuition

When writing his early works, including his 1907 essay and 1910 book, Cassirer had just absorbed the lessons of Russell, Frege, and Dedekind. By the 1920s-30s, he has become familiar with further developments in the foundations of mathematics, including works by Poincaré, Brouwer, and Weyl in which "intuition" is brought back in seemingly essential ways. Their works are highly controversial, however, as they lead to an radical "intuitionist" revision of modern mathematics. Cassirer is no friend of such a revision; he wants to hold on to modern mathematics in its classic, "function-oriented" form. Nevertheless, a rethinking of his Dedekindian logicism seems to occur at this point, probably prompted, at least in part, by the intuitionists' challenge. Cassirer also develops his more general position further during this period, resulting in his mature "philosophy of symbolic forms". And intuition is reemphasized in it as well, also in fundamental ways, as we want to suggest now.

Cassirer's philosophy of symbolic forms is a complex, multi-faceted topic; here we can only address one aspect directly relevant for our purposes. At this later, more mature stage of his career, Cassirer is no longer focused on mathematical and scientific knowledge alone, as he was earlier. Instead, he has extended the reach of his approach so as to also cover mythical thought, religion, ordinary language, art, history, etc. All of these are ways of "objectifying", or of "world making", that deserve philosophical attention. Cassirer argues, in fact, that such "symbolic forms" are all inter-related, even inter-dependent, in the end. Moreover, both myth and ordinary language, the novel forms he explores in most detail, are deeply rooted in sensory, hence intuitive, relations to the world. But then, mathematical science, as another "symbolic form", needs to be re-thought too. And this includes Dedekind's conception of number, which continues to serve as Cassirer's paradigm for modern "function-based" knowledge. More specifically, that conception now needs to be seen as growing out of more rudimentary numerical ideas, as embodied in ordinary language and our pre-scientific, even mythical immersion in the world.[31]

In addition, Cassirer comes to accept the following general point during this period: A central aspect of all symbolic forms, insofar as they lead to "objectification", is that "objects" are constituted in them in certain ways. Actually, this concerns not just "objects", but also "subjects". Moreover, Cassirer is interested in how the very distinction between subject and object emerges in the process. For myth and ordinary language, this involves sensory intuition, in interestingly different ways. Modern science, including pure mathematics, involves "objects" as well, but in a "thinner" sense. Even a theory like arithmetic, insofar as it has content (Cassirer is no mere formalist), involves them. And this is so in two related ways: On the one hand, there are the structuralist "objects" resulting from Dedekind abstraction. On the other hand, arithmetic involves a prior "existential" and "constructive" aspect via Dedekind's logical constructions, e.g., of an initial simple infinity. Now, at point

[31] So far there is very little literature on this topic; cf. Heis (2015) as an exception.

Cassirer talks about these as involving a kind of "positing". But how is such "positing" supposed to work? And again, what is its relation to "intuition"?

Once again, from Cassirer's 1907 essay and 1910 book it is easy to get the sense that the positing at issue does not require "intuition" at all. Or more cautiously now, it does not do so in the sense of requiring intuitive geometric constructions à la Euclid (of $\sqrt{2}$ as the diagonal of a square etc.). Does it require "intuition" in a secondary, dependent sense, involving space or time in some "concrete" way? Here we noted a tension in the previous section. Pushing this issue further now, might it require intuition in an additional, more "abstract" sense too? What Cassirer seems to realize during this period, in his study of the various symbolic forms, is the following: All symbolic forms, exactly insofar as they lead to "objectification", involve a kind of "orientation" and "individuation", starting with a distinction between subject/self and objects outside of it (then also a sequential order in thought connected to changes in the external objects etc.). Typically this involves sensory intuition, where the constituted objects are situated relative to the subject in space and time, as the cases of myth and ordinary language illustrate most clearly. But such "orientation" and "individuation" plays a role more generally, in all symbolic forms. And from a Kantian point of view, one might think of it as an abstract kind of "intuition".[32]

If we now turn back to modern mathematics, thought of along Dedekindian lines, the question arises: Does such abstract "intuition" play a role here too; or can we proceed "purely conceptually" in this context?[33] A first reason to deny the latter derives from Cassirer's view that all symbolic forms are inter-related. Thus, modern mathematics shouldn't be seen as totally separate from ordinary, and even mythical, uses of number terms, themselves grounded in intuition. Second, arguably the very notion of "object", including the "thin" objects of structuralist mathematics, cannot

[32] Note Cassirer's repeated references to Kant's essay, "Was heißt sich im Denken orientieren?" (Kant 1786). As they indicate, he comes to see this as an original Kantian point. There are also deep connections to Kant's understanding of his "Copernican Revolution" and of the process of "schematization" in cognition; cf. Keller (2015).

[33] Frege, for one, argues that we can. But for Cassirer, Frege's (and Russell's) argument is flawed since it relies on an outdated and misleading "substance-oriented" notion of class.

be understood apart from its origin in pre-scientific, more intuitive thinking. But there is also a third, more direct argument available. As we saw, Dedekind's use of abstraction, which underwrites his structuralism, is preceded by the construction of other systems of objects, both in the case of the natural and the real numbers. Crucially, these constructions implicitly rely on the distinction between a subject/self and individual objects external to it. (Remember Dedekind's reliance on his "self" in the proof of Theorem 66; but his appeal to a series of "thoughts", distinct both from it and each other, matters too.) [34] Moreover, we are back to the "existential" aspect of basic mathematical concepts here, thus to the "synthetic" nature of mathematics. Finally, this might also be where the "concrete realization" of mathematical concepts, already acknowledged in Cassirer's early writings, can find a home, i.e., their previously obscure role might thus be clarified.

It is hard to single out one passage, or a few passages, in Cassirer's later writings where he makes any of these arguments explicitly, although they are not far below the surface at various points, we believe, especially in *Erkenntnisproblem*, *Vol. IV*. [35] However, what has then happened to "logicism", seemingly endorsed by Cassirer throughout? Here our final suggestion is this: Acknowledging the sort of intuition just considered—involving basic "orientation", "individuation", etc.—is not necessarily incompatible with the core of Dedekindian logicism. One can continue to hold that all theorems about the natural numbers can, and should, be deduced from the concept of simple infinity alone; similarly for the real numbers and the concept of a complete order field. More particularly, one does not need to appeal to intuitive evidence in the traditional Euclidean sense in those deductions (e.g., of the Mean Value Theorem); nor do we need to appeal to intuition as a psychological process at that level. At the same time, basic mathematical concepts, such as those of a simple infinity and complete ordered field, are now seen as grounded in

[34] Note that the intuition required here is minimal. Thus, the use of signs like '∅' counts as well. This concerns the variant of Dedekind's approach considered in Section 3.

[35] Cf. Cassirer (1950), Ch. 4, especially where Cassirer comments sympathetically on Poincaré's appeal to intuition in mathematics, although the passages remain ambiguous.

intuition with respect to their "existential import".[36] Hence we end up with a more mitigated, less exclusionary form of "logicism" than might have appeared earlier.

SUMMARY AND CONCLUSION

The line of thought presented in the present essay can be summarized as follows. In Sections 1-3, an interpretation of Dedekind's logicist position was presented. It constitutes a version of structuralist logicism, or logicist structuralism, that, as such, is significantly different from Frege's and Russell's forms of logicism. After introducing and motivating it, some influential objections against Dedekind's views were mentioned, and he was defended against them in part, especially against the dismissal of his approach as involving a crude, subjective form of psychologism. It was granted, however, that his presentation contains some noteworthy gaps, especially insofar as he does not explicitly formulate the basic principles or laws needed to ground his procedures, as Frege noted early on. We then considered some suggestions for how to fill those gaps on his behalf. But these were sketchy, and questions about whether the results would still be "logicist" remained.

In Section 4, our discussion turned to Cassirer, especially Cassirer's initial reception of Dedekind's logicism. Cassirer was not only a very perceptive reader of Dedekind; he also attempted to combine Dedekind's position with basic Kantian assumptions, e.g. about the role of the "construction of concepts" in mathematics. Moreover, that project was both rooted in the "logical idealism" of Marburg Neo-Kantianism and distinctive in certain ways, as was elaborated further in Section 5. Characteristic for the early Cassirer, on which the discussion up to this point focused, are the following points: his embrace of Russellian relational logic; his defense of Dedekind's structuralist logicism; and his effort to embed both in a historical account of the shift from "substance concepts" to "function concepts" in modern science, in such a way that Dedekind's contributions can be seen as paradigmatic for the latter.

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[36] A remaining question is what is involved in this grounding etc., from a mathematical point of view. If not full Euclidean geometry, then perhaps some topological features? Clearly this topic deserves more attention that we can give it here.

In Section 6, finally, a brief foray into Cassirer's later, more mature views about mathematics, as part of his broader philosophy of symbolic forms, was offered. We did not provide a general discussion of that position, only a few observations about how his views about intuition developed or got clarified in it. If our suggestions at the end are on the right track, what one ends up with is a form of Dedekindian logicism tempered by a subtle appeal to intuition. The result is a development of Marburg Neo-Kantianism, but also a return to an originally Kantian insight, namely a conception to intuition that concerns basic orientation and individuation. Overall, Cassirer's approach, especially in his mature writings, reveals itself as a distinctive attempt to synthesize logicism with the Kantian heritage. That is where the paper had to stop, leaving a further defense and more detailed elaboration of the suggested perspective on the mature Cassirer for other occasions.

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